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Exact solution of N directed non-intersecting walks interacting with one or two boundaries

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Abstract. The partition function for the problem of an arbitrary number of directed non-intersecting walks interacting with one or two walls parallel to the direction of the walks is calculated exactly utilizing a theorem recently proved concerning the Bethe ansatz for the eigenvectors of the transfer matrix of the five-vertex model. This theorem shows that the completeness of the Bethe ansatz eigenvectors for the N-walk transfer matrix can be deduced from the completeness of the one-walk eigenvectors.

1. Introduction

Recently, the problems of one and two directed walks interacting with one and two walls via contact interactions on the square lattice have been solved exactly [1]. In particular, the partition functions for fixed length and fixed starting and ending positions have been evaluated. Another recent development [2] has been the proof of two theorems concerning the evaluation of the partition function of N such walks with arbitrary inhomogeneous (with respect to the direction perpendicular to the directness) weights: these theorems give the answer in terms of a determinant *provided* that the solution of the one-walk problem can be structured in a particular fashion. In this paper we show that the one-walk solution for the case of surface contact interactions with homogeneous bulk weights satisfies the conditions of those theorems, which we write as a single theorem for the purposes of this paper, and hence we give the solution of N walks in a strip and in a half-plane interacting with their surfaces. In the case of the strip, one boundary interaction is dependent on the other. The ideas behind this method of solution originated in the mapping from the five-vertex model (a sub-case of the more well known six-vertex model) to N non-intersecting walks [3, 4]. It should be noted that with homogeneous weights away from the boundaries but extra weights at the boundaries the six-vertex model has been considered previously [5]. However, in [5] and most other studies of the six-vertex model, only such properties of the model are calculated that are averages over all numbers of walks, N. Here, in contrast, we find the partition function for a fixed number of walks, N, of a fixed finite length t. Our formulae generalize the 'master formulae' of Fisher (equation (5.9) of [6]) and of Forrester (equation (4) of [7]). Physically, multiple walks near a sticky wall are a simple model of the adsorption of a polymer network.

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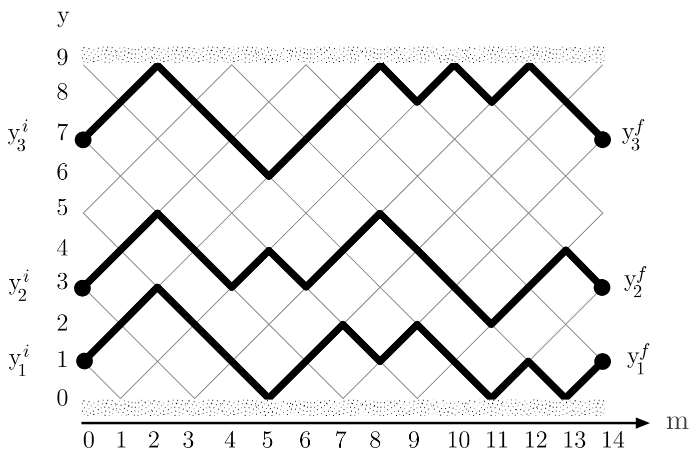

Figure 1. Three non-intersecting directed walks of length $t = 14$ in a strip of width $L = 9$. The variables y^i, y^f and t are shown. The walk closest to the lower wall has weight $v(1)w(1,2)w(2,3)w(3,2)\ldots w(1,0)w(0,1) = \kappa^3$ with $y_1^i = 1$ and $y_1^f = 1$.

2. The model

A lattice path or walk in this paper is a walk on a square lattice rotated through 45° which has steps in only the north-east or south-east directions, and with sites labelled (m, y) (see figure 1). A set of walks is *non-intersecting* if they have no sites in common. We are concerned with enumerating the number of configurations of N non-intersecting walks, starting and ending at given positions, in two geometries: (1) walks which are confined to the upper half-plane; and (2) walks which are confined to a strip of a given width, L. In particular, we are interested in interacting cases where the walks nearest the boundaries are attracted or repulsed by contact interactions. In this paper we shall focus on case (2) since the other case can be easily derived from it.

First, we describe the model considered by Brak *et al* [2]: this requires the following sub-domains of $\mathbb{Z}^N$:

$$\overset{o}{\mathcal{S}}_L = \{y \mid 1 \leqslant y \leqslant L,\ y \in \mathbb{Z} \text{ and } y \text{ odd}\} \tag{1a}$$

$$\overset{e}{\mathcal{S}}_L = \{y \mid 0 \leqslant y \leqslant L,\ y \in \mathbb{Z} \text{ and } y \text{ even}\} \tag{1b}$$

$$\mathcal{S}_L = \{y \mid 0 \leqslant y \leqslant L,\ y \in \mathbb{Z}\} \tag{1c}$$

$$\overset{o}{\mathcal{U}}_L = \{(y_1, \ldots, y_N) \mid 1 \leqslant y_1 < \cdots < y_N \leqslant L,\ y_i \in \overset{o}{\mathcal{S}}_L\} \tag{1d}$$

$$\overset{e}{\mathcal{U}}_L = \{(y_1, \ldots, y_N) \mid 0 \leqslant y_1 < \cdots < y_N \leqslant L,\ y_i \in \overset{e}{\mathcal{S}}_L\} \tag{1e}$$

$$\mathcal{U}_L = \{(y_1, \ldots, y_N) \mid 0 \leqslant y_1 < \cdots < y_N \leqslant L,\ y_i \in \mathcal{S}_L\}. \tag{1f}$$

We will use $\overset{p}{\mathcal{U}}_L$ to denote $\overset{o}{\mathcal{U}}_L$ or $\overset{e}{\mathcal{U}}_L$. N non-intersecting walks were considered, confined to a strip of width L, such that they started at y-coordinates $\boldsymbol{y}^i = (y_1^i, \ldots, y_N^i) \in \overset{p}{\mathcal{U}}_L$ in column $m = 0$ of the lattice sites and terminated after t steps at y-coordinates $\boldsymbol{y}^f = (y_1^f, \ldots, y_N^f) \in \overset{p'}{\mathcal{U}}_L$ in the tth column. If t is even then $p' = p$ or else $p' = \bar{p}$, where $\bar{p}$ is the opposite parity to p. The strip width L was considered to be odd only so that $|\ \overset{e}{\mathcal{U}}_L\ | = |\ \overset{o}{\mathcal{U}}_L\ | = \binom{\frac{1}{2}(L+1)}{N}$: we do the same in this work. Paths were considered such that if $(m-1, y)$, with $y \in \mathcal{S}_L$, is the position of a path in column $m-1$ the only possible positions for that path in column m are (m, y')

with $y' = y \pm 1$ and $y' \in \mathcal{S}_L$. The non-intersection is defined through the constraint that if there are N distinct sites occupied at $m = 0$ then in each column of sites ($0 \leqslant m \leqslant t$) there are exactly N occupied sites. Hence the y-coordinates of the occupied sites in any column are restricted to satisfy $\boldsymbol{y} \in \overset{p}{\mathcal{U}}_L$ with the parity p depending on whether m is even or odd. The walk problem associated with the five-vertex problem [8, 4] was generalized by Brak *et al* [2] by the assigning of a weight $w(y, y')$ to the lattice edge from site $(m-1, y)$ to (m, y') with $y' = y \pm 1$ (see figure 1). Notice that, since $w(y, y')$ is assumed independent of the column index t, due to the square lattice structure the weights are periodic in the t direction with period two: note that if $y \in \overset{p}{\mathcal{S}}_L$ then $y' \in \overset{\bar{p}}{\mathcal{S}}_L$, and in general $w(y, y') \neq w(y', y)$. An arbitrary weight $v(y^i)$ was also associated with each of the sites occupied at $m = 0$. The weight associated with a given set of walks is the product of w weights over all edges occupied by the walks multiplied by the product of the v weights for each of the initial sites occupied. The partition function, $\overset{=}{Z}{}_t^{\mathcal{N}}(\boldsymbol{y}^i \to \boldsymbol{y}^f)$, of N walks of length t starting at $\boldsymbol{y} = \boldsymbol{y}^i$ in column $m = 0$ and finishing at $\boldsymbol{y} = \boldsymbol{y}^f$ in column $m = t$ is the sum of these weights over all sets of walks connecting $\boldsymbol{y}^i$ and $\boldsymbol{y}^f$:

$$\overset{=}{Z}{}_t^{\mathcal{N}}(\boldsymbol{y}^i \to \boldsymbol{y}^f) = \sum_{\mathcal{Y}} \prod_{j=1}^{N} v(y_j(0)) \prod_{m=1}^{t} w(y_j(m-1), y_j(m)) \tag{2}$$

where $y_j(m)$ is the position of the jth walk in column m and the set $\mathcal{Y}$ is given by

$$\begin{aligned}\mathcal{Y} = \{y_j(m) | 1 \leqslant j \leqslant N, 0 \leqslant m \leqslant t, 1 \leqslant y_1(m) < y_2(m) < \cdots < y_N(m) \leqslant L \\ y_j(m) = y_j(m-1) \pm 1 \text{ and } y_j(0) = y_j^i, y_j(t) = y_j^f\}.\end{aligned} \tag{3}$$

When $N = 1$ we denote this partition function as $\overset{=}{Z}{}_t^{\mathcal{S}}(y^i \to y^f)$.

In this paper we shall only consider the case L odd and for $y \in \mathcal{S}_L$

$$v(y) = \begin{cases} \kappa & \text{for} \quad y = 0 \\ \mu = 1 + \dfrac{1}{(\kappa - 1)} & \text{for} \quad y = L \\ 1 & \text{otherwise} \end{cases} \tag{4}$$

and

$$w(y, y') = v(y') \qquad \text{for} \quad y, y' \in \mathcal{S}_L \ . \tag{5}$$

It should be noted that while we have a model where weights are associated with edges the model chosen is readily seen to be exactly equivalent to one where a weight κ is associated with every occupied site at $y = 0$ and a weight μ is associated with every occupied site at $y = L$. A $(1, 0)$ step weight is 'transferred' to a $y = 0$ site weight and a $(L-1, L)$ step weight is transferred to a $y = L$ site weight with the initial weights $v(0)$ and $v(L)$ taking care of the first site of the paths possibly in contact with either of the boundaries (i.e. if $y_1^i = 0$ or $y_N^i = L$ respectively).

In the limit $L \to \infty$ with $\boldsymbol{y}^i$ and $\boldsymbol{y}^f$ fixed and finite, the partition function for N walks near a single wall is obtained

$$\bar{Z}_t^{\mathcal{N}}(\boldsymbol{y}^i \to \mathbf{y}^f) = \lim_{L \to \infty} \overset{=}{Z}{}_t^{\mathcal{N}}(\boldsymbol{y}^i \to \boldsymbol{y}^f). \tag{6}$$

When $N = 1$ we denote the one-wall partition function as $\bar{Z}_t^{\mathcal{S}}(y^i \to y^f)$.

3. BEO theorem

We summarize the work of Brak *et al* [2] in one theorem, which we refer to as the BEO theorem. First, we define the requisite transfer matrices.

Transfer matrix definitions. *Let* $y \in \overset{e}{\mathcal{S}}_L$ *and* $y' \in \overset{o}{\mathcal{S}}_L$. *For* $N = 1$ *the one-step transfer matrices are defined as*

$$(\overset{eo}{\boldsymbol{T}}_1)_{y,y'} = \begin{cases} 0 & \text{if} \quad |y - y'| > 1 \\ w(y, y') & \text{if} \quad |y - y'| = 1 \end{cases} \tag{7a}$$

and

$$(\overset{oe}{\boldsymbol{T}}_1)_{y',y} = \begin{cases} 0 & \text{if} \quad |y' - y| > 1 \\ w(y', y) & \text{if} \quad |y - y'| = 1. \end{cases} \tag{7b}$$

The N*-walk one-step transfer matrices for* $N > 1$ *are constructed from sub-matrices of a direct product of the above* $N = 1$ *matrices:*

$$(\overset{oe}{\boldsymbol{T}}_N)_{\boldsymbol{y},\boldsymbol{y}'} = \left(\bigotimes_{i=1}^{N} \overset{oe}{\boldsymbol{T}}_1 \right)_{\boldsymbol{y},\boldsymbol{y}'} \qquad \boldsymbol{y} \in \overset{o}{\mathcal{U}}_L \quad \text{and} \quad \boldsymbol{y}' \in \overset{e}{\mathcal{U}}_L \tag{8a}$$

and

$$(\overset{eo}{\boldsymbol{T}}_N)_{\boldsymbol{y}',\boldsymbol{y}} = \left(\bigotimes_{i=1}^{N} \overset{eo}{\boldsymbol{T}}_1 \right)_{\boldsymbol{y}',\boldsymbol{y}} \qquad \boldsymbol{y}' \in \overset{e}{\mathcal{U}}_L \quad \text{and} \quad \boldsymbol{y} \in \overset{o}{\mathcal{U}}_L. \tag{8b}$$

The two-step transfer matrices for all N *are defined as*

$$\overset{e\cdot e}{\boldsymbol{T}}_N = \overset{eo}{\boldsymbol{T}}_N \overset{oe}{\boldsymbol{T}}_N \tag{9a}$$

$$\overset{o\cdot o}{\boldsymbol{T}}_N = \overset{oe}{\boldsymbol{T}}_N \overset{eo}{\boldsymbol{T}}_N. \tag{9b}$$

In the normal fashion the N-walk transfer matrices can be used to calculate the N-walk partition functions. The BEO theorem describes under what conditions on the $N = 1$ transfer matrices the partition function for N walks can be computed from these matrices using a Bethe ansatz.

BEO theorem. *Let* $\{\overset{o}{\varphi}{}_k^{\mathrm{R}}\}_{k\in\mathcal{K}_1}$ *and* $\{\overset{e}{\varphi}{}_k^{\mathrm{R}}\}_{k\in\mathcal{K}_1}$, *where* $\mathcal{K}_1$ *is some index set, be maximal sets of independent vectors satisfying*

$$\overset{eo}{\boldsymbol{T}}_1 \overset{o}{\varphi}{}_k^{\mathrm{R}} = \lambda_k \, \overset{e}{\varphi}{}_k^{\mathrm{R}} \qquad \text{and} \qquad \overset{oe}{\boldsymbol{T}}_1 \overset{e}{\varphi}{}_k^{\mathrm{R}} = \lambda_k \, \overset{o}{\varphi}{}_k^{\mathrm{R}} \tag{10}$$

with $\lambda_k \in \mathbb{C}$. *Let* $\overset{e\cdot e}{\boldsymbol{T}}_N$ *and* $\overset{o\cdot o}{\boldsymbol{T}}_N$ *for all* N *be given by equations (9). By imposing an arbitrary ordering on the elements of* $\mathcal{K}_1$ *define*

$$\mathcal{K}_N = \{\boldsymbol{k} = (k_1, k_2, \ldots, k_N) | k_i \in \mathcal{K}_1 \text{ and } k_1 < k_2 < \cdots < k_N\} \tag{11}$$

and

$$\Lambda_{\boldsymbol{k}} = \prod_{\alpha=1}^{N} \lambda_{k_\alpha}. \tag{12}$$

(a) *If the set* $\mathcal{K}_1$ *is non-empty then* $\overset{o}{\varphi}{}_k^{\mathrm{R}}$ *and* $\overset{e}{\varphi}{}_k^{\mathrm{R}}$ *are right eigenvectors of* $\overset{o\cdot o}{\boldsymbol{T}}_1$ *and* $\overset{e\cdot e}{\boldsymbol{T}}_1$ *respectively with eigenvalue* λ_k^2.

(b) *If for* $p \in \{e, o\}$ *the sets* $\{\overset{p}{\varphi}{}^{\mathrm{R}}_k\}_{k\in\mathcal{K}_1}$ *span the column spaces of* $\overset{e\cdot e}{\boldsymbol{T}}_1$ *and* $\overset{o\cdot o}{\boldsymbol{T}}_1$ *respectively, then:*

(i) *corresponding sets* $\{\overset{o}{\varphi}{}^{\mathrm{L}}_k\}_{k\in\mathcal{K}_1}$ *and* $\{\overset{e}{\varphi}{}^{\mathrm{L}}_k\}_{k\in\mathcal{K}_1}$ *of row vectors may be found such that*

$$\overset{p}{\varphi}{}^{\mathrm{L}*}_k \cdot \overset{p}{\varphi}{}^{\mathrm{R}}_{k'} = \delta_{k,k'} \qquad \textit{and} \qquad \sum_{k\in\mathcal{K}_1} \overset{p}{\varphi}{}^{\mathrm{R}}_k(y)\, \overset{p}{\varphi}{}^{\mathrm{L}*}_k(y') = \delta_{y,y'} \tag{13}$$

for each $p \in \{e, o\}$*, where the* $*$ *denotes complex conjugation, and*

$$\overset{o}{\varphi}{}^{\mathrm{L}}_k\, \overset{oe}{\boldsymbol{T}}_1 = \lambda_k\, \overset{e}{\varphi}{}^{\mathrm{L}}_k \qquad \textit{and} \qquad \overset{e}{\varphi}{}^{\mathrm{L}}_k\, \overset{eo}{\boldsymbol{T}}_1 = \lambda_k\, \overset{o}{\varphi}{}^{\mathrm{L}}_k \tag{14}$$

and further $\overset{o}{\varphi}{}^{\mathrm{L}}_k$ *and* $\overset{e}{\varphi}{}^{\mathrm{L}}_k$ *are left eigenvectors of* $\overset{o\cdot o}{\boldsymbol{T}}_1$ *and* $\overset{e\cdot e}{\boldsymbol{T}}_1$ *respectively with eigenvalue* λ_k^2*;*

(ii) *for* $C \in \{L, R\}$ *and* $p \in \{e, o\}$ *the vectors* $\{\overset{p}{\Phi}{}^{\mathrm{C}}_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathcal{K}_N}$ *given by the Bethe ansatz,*

$$\overset{p}{\Phi}{}^{\mathrm{C}}_{\boldsymbol{k}}(\boldsymbol{y}) = \sum_{\sigma\in P_N} \epsilon_\sigma \prod_{\alpha=1}^{N} \overset{p}{\varphi}{}^{\mathrm{C}}_{k_{\sigma_\alpha}}(y_\alpha) = \sum_{\sigma\in P_N} \epsilon_\sigma \prod_{\alpha=1}^{N} \overset{p}{\varphi}{}^{\mathrm{C}}_{k_\alpha}(y_{\sigma_\alpha}) \qquad \boldsymbol{y} \in \overset{p}{\mathcal{U}}_L \tag{15}$$

where P_N *is the set of* $N!$ *permutations of* $\{1, 2, \ldots, N\}$, $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_N) \in P_N$ *and* ϵ_σ *is the signature of the permutation* σ*, satisfy*

$$\overset{oe}{\boldsymbol{T}}_N\, \overset{e}{\Phi}{}^{\mathrm{R}}_{\boldsymbol{k}} = \Lambda_{\boldsymbol{k}}\, \overset{o}{\Phi}{}^{\mathrm{R}}_{\boldsymbol{k}} \qquad \textit{and} \qquad \overset{eo}{\boldsymbol{T}}_N\, \overset{o}{\Phi}{}^{\mathrm{R}}_{\boldsymbol{k}} = \Lambda_{\boldsymbol{k}}\, \overset{e}{\Phi}{}^{\mathrm{R}}_{\boldsymbol{k}} \tag{16}$$

and

$$\overset{o}{\Phi}{}^{\mathrm{L}}_{\boldsymbol{k}} \overset{oe}{\boldsymbol{T}}_N = \overset{e}{\Phi}{}^{\mathrm{L}}_{\boldsymbol{k}} \Lambda_{\boldsymbol{k}} \qquad \overset{e}{\Phi}{}^{\mathrm{L}}_{\boldsymbol{k}} \overset{eo}{\boldsymbol{T}}_N = \overset{o}{\Phi}{}^{\mathrm{L}}_{\boldsymbol{k}} \Lambda_{\boldsymbol{k}} \tag{17}$$

(iii) *the sets* $\{\overset{p}{\Phi}{}^{\mathrm{C}}_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathcal{K}_N}$ *given by (15) are maximal sets of independent vectors that span the column* ($C = R$) *and row* ($C = L$) *spaces of* $\overset{o\cdot o}{\boldsymbol{T}}_N$ ($p = o$) *and* $\overset{e\cdot e}{\boldsymbol{T}}_N$ ($p = e$)*;*

(iv) *the* N*-walk partition function is given by*

$$\overset{=}{Z}{}^{\mathcal{N}}_t(\boldsymbol{y}^i \to \boldsymbol{y}^f) = \prod_{\alpha=1}^{N} v(y^i_\alpha) \sum_{\boldsymbol{k}\in\mathcal{K}_N} \overset{p'}{\Phi}{}^{\mathrm{R}}_{\boldsymbol{k}}(\boldsymbol{y}^i)\Lambda^t_{\boldsymbol{k}}\, \overset{p}{\Phi}{}^{\mathrm{L}*}_{\boldsymbol{k}}(\boldsymbol{y}^f)$$

$$\boldsymbol{y}^i \in \overset{p'}{\mathcal{U}}_L \quad \textit{and} \quad \boldsymbol{y}^f \in \overset{p}{\mathcal{U}}_L \tag{18}$$

where if t *is even,* $p' = p$ *but otherwise* p *and* p' *are of opposite parity, and*

(v)
$$\overset{=}{Z}{}^{\mathcal{N}}_t(\boldsymbol{y}^i \to \mathbf{y}^f) = \det \|\overset{=}{Z}{}^{\mathcal{S}}_t(y^i_\alpha \to y^f_\beta)\|_{\alpha,\beta=1,\ldots,N}. \tag{19}$$

It was shown also that if $L > (t - |y^i_N - y^f_N|)/2 + \max(y^i_N, y^f_N)$ then

$$\overset{=}{Z}{}^{\mathcal{S}}_t(y^i_\alpha \to y^f_\beta) = \bar{Z}^{\mathcal{S}}_t(y^i_\alpha \to y^f_\beta) \tag{20}$$

and hence

$$\bar{Z}^{\mathcal{N}}_t(\boldsymbol{y}^i \to \mathbf{y}^f) = \det \|\bar{Z}^{\mathcal{S}}_t(y^i_\alpha \to y^f_\beta)\|_{\alpha,\beta=1,\ldots,N}. \tag{21}$$

4. Results

In the next section we demonstrate that with the weights given by (5) the $N = 1$ transfer matrices satisfy the conditions of the BEO theorem and hence show that the partition function,

$Z_t^{\mathcal{N}}(\boldsymbol{y}^i \to \boldsymbol{y}^f)$, of N non-intersecting walks with weights (5) and (4) for which the jth walk starts at y_j^i and arrives at y_j^f after t steps is given by the following determinant:

$$Z_t^{\mathcal{N}}(\mathbf{y}^i \to \mathbf{y}^f) = \begin{vmatrix} Z_t^{\mathcal{S}}(y_1^i \to y_1^f) & Z_t^{\mathcal{S}}(y_1^i \to y_2^f) & \dots & Z_t^{\mathcal{S}}(y_1^i \to y_N^f) \\ Z_t^{\mathcal{S}}(y_2^i \to y_1^f) & Z_t^{\mathcal{S}}(y_2^i \to y_2^f) & \dots & Z_t^{\mathcal{S}}(y_2^i \to y_N^f) \\ \vdots & \vdots & \vdots & \vdots \\ Z_t^{\mathcal{S}}(y_N^i \to y_1^f) & Z_t^{\mathcal{S}}(y_N^i \to y_2^f) & \dots & Z_t^{\mathcal{S}}(y_N^i \to y_N^f) \end{vmatrix} \tag{22}$$

where $Z_t^{\mathcal{S}}(y_j^i \to y_k^f)$ is the generating function for configurations of a single walk starting at y_j^i and ending at y_k^f in the presence of a wall or two walls. In the former case, with $\bar{\kappa} = \kappa - 1$, $Z_t^{\mathcal{S}}(y^i \to y^f)$ is replaced by

$$\begin{aligned} \bar{Z}_t^{\mathcal{S}}(y^i \to y^f) = & \binom{t}{\frac{1}{2}(t - y^i + y^f)} - \binom{t}{\frac{1}{2}(t - y^i - y^f - 2)} \\ & + \sum_{n \geqslant 1} \bar{\kappa}^n \left\{ \binom{t}{\frac{1}{2}(t - 2n - y^i - y^f + 2)} - \binom{t}{\frac{1}{2}(t - 2n - y^i - y^f - 2)} \right\}. \end{aligned} \tag{23}$$

When there are two walls a width $L > 0$ apart $Z_t^{\mathcal{S}}(y^i \to y^f)$ must be replaced by

$$\begin{aligned} \overset{=}{Z}{}_t^{\mathcal{S}}(y^i \to y^f) = & \frac{(\kappa - 2)\kappa^{t+1}}{\bar{\kappa}^{\frac{1}{2}(t+y^i+y^f+2)}\{1 - \bar{\kappa}^{-L}\}} \\ & + \sum_{m \in \mathbb{Z}} \left\{ \binom{t}{\frac{1}{2}(t - y^i + y^f) + Lm} - \binom{t}{\frac{1}{2}(t - y^i - y^f - 2) + Lm} \right. \\ & + \sum_{n \geqslant 1} \bar{\kappa}^n \left[\binom{t}{\frac{1}{2}(t - 2n - y^i - y^f + 2) + Lm} \right. \\ & \left. \left. - \binom{t}{\frac{1}{2}(t - 2n - y^i - y^f - 2) + Lm} \right] \right\} \end{aligned} \tag{24}$$

which is not a polynomial in κ, but rather a Laurent polynomial in $\bar{\kappa}$.

5. Solution of N non-intersecting walks interacting with a wall

To solve the N-walk problem we establish that the conditions of the BEO theorem hold. To accomplish this we must examine the one-walk case. We point out that we examine the one-walk problem not to solve this problem per se, as it has been solved previously [1] using a different method, but rather to establish the technical conditions of the BEO theorem. Once we have demonstrated that the one-walk transfer matrix problem satisfies the conditions of the BEO theorem the determinantal result (22) for the partition function for N walks then follows from the conclusions of the theorem.

5.1. *Establishing the conditions of the BEO theorems*

In order to use the BEO theorem we first need a set of one-walker left and right eigenvectors of $\overset{e\cdot e}{\boldsymbol{T}}_1$ and $\overset{o\cdot o}{\boldsymbol{T}}_1$ that span the row and column spaces respectively of these matrices. To do this we consider the equations (10) which allows us to use part (a) of the theorem to establish the conditions of part (b).

For the right vectors, $\overset{e}{\varphi}{}^{\mathrm{R}}_k$, we first solve the sets of equations: $\overset{eo}{\boldsymbol{T}}_1 \overset{o}{\varphi}{}^{\mathrm{R}}_k = \lambda\, \overset{e}{\varphi}{}^{\mathrm{R}}_k$ meaning

$$\overset{o}{\varphi}{}^{\mathrm{R}}_k(1) = \lambda\, \overset{e}{\varphi}{}^{\mathrm{R}}_k(0) \tag{25}$$

$$\overset{o}{\varphi}{}^{\mathrm{R}}_k(y) + \overset{o}{\varphi}{}^{\mathrm{R}}_k(y+2) = \lambda\, \overset{e}{\varphi}{}^{\mathrm{R}}_k(y+1) \qquad 1 \leqslant y \leqslant L-4 \quad \text{and} \quad y \in \overset{o}{\mathcal{S}}_L \tag{26}$$

$$\overset{o}{\varphi}{}^{\mathrm{R}}_k(L-2) + \mu\, \overset{o}{\varphi}{}^{\mathrm{R}}_k(L) = \lambda\, \overset{e}{\varphi}{}^{\mathrm{R}}_k(L-1) \tag{27}$$

and $\overset{oe}{\boldsymbol{T}}_1 \overset{e}{\varphi}{}^{\mathrm{R}}_k = \lambda\, \overset{o}{\varphi}{}^{\mathrm{R}}_k$ meaning

$$\kappa\, \overset{e}{\varphi}{}^{\mathrm{R}}_k(0) + \overset{e}{\varphi}{}^{\mathrm{R}}_k(2) = \lambda\, \overset{o}{\varphi}{}^{\mathrm{R}}_k(1) \tag{28}$$

$$\overset{e}{\varphi}{}^{\mathrm{R}}_k(y) + \overset{e}{\varphi}{}^{\mathrm{R}}_k(y+2) = \lambda\, \overset{o}{\varphi}{}^{\mathrm{R}}_k(y+1) \qquad 2 \leqslant y \leqslant L-3 \quad \text{and} \quad y \in \overset{e}{\mathcal{S}}_L \tag{29}$$

$$\overset{e}{\varphi}{}^{\mathrm{R}}_k(L-1) = \lambda\, \overset{o}{\varphi}{}^{\mathrm{R}}_k(L). \tag{30}$$

Once we have found the right vectors the BEO theorem guarantees us solutions for the left vectors which satisfy the following equations (14): that is, $\overset{e}{\varphi}{}^{\mathrm{L}}_k\, \overset{eo}{\boldsymbol{T}}_1 = \lambda\, \overset{o}{\varphi}{}^{\mathrm{L}}_k$ meaning

$$\overset{e}{\varphi}{}^{\mathrm{L}}_k(y) + \overset{e}{\varphi}{}^{\mathrm{L}}_k(y+2) = \lambda\, \overset{o}{\varphi}{}^{\mathrm{L}}_k(y+1) \qquad 0 \leqslant y \leqslant L-3 \quad \text{and} \quad y \in \overset{e}{\mathcal{S}}_L \tag{31}$$

$$\mu\, \overset{e}{\varphi}{}^{\mathrm{L}}_k(L-1) = \lambda\, \overset{o}{\varphi}{}^{\mathrm{L}}_k(L) \tag{32}$$

and $\overset{o}{\varphi}{}^{\mathrm{L}}_k\, \overset{oe}{\boldsymbol{T}}_1 = \lambda\, \overset{e}{\varphi}{}^{\mathrm{L}}_k$ meaning

$$\kappa\, \overset{o}{\varphi}{}^{\mathrm{L}}_k(1) = \lambda\, \overset{e}{\varphi}{}^{\mathrm{L}}_k(0) \tag{33}$$

$$\overset{o}{\varphi}{}^{\mathrm{L}}_k(y) + \overset{o}{\varphi}{}^{\mathrm{L}}_k(y+2) = \lambda\, \overset{e}{\varphi}{}^{\mathrm{L}}_k(y+1) \qquad 1 \leqslant y \leqslant L-2 \quad \text{and} \quad y \in \overset{e}{\mathcal{S}}_L\ . \tag{34}$$

We only consider the special case $\kappa - 1 = 1/(\mu - 1)$ for which the equations above can be explicitly solved to give the following. For $y \in \overset{p}{\mathcal{S}}_L$

$$\overset{p}{\varphi}{}^{\mathrm{L}}_k(y) = \varphi_k(y) \tag{35}$$

$$\overset{p}{\varphi}{}^{\mathrm{R}}_k(y) = \varphi_k(y)/v(y) \tag{36}$$

where

$$\varphi_k(y) = \begin{cases} \left[\dfrac{\mu(\kappa-2)}{1-(\mu-1)^L}\right]^{1/2} (\kappa-1)^{-y/2} & \text{if} \quad k = 0 \\ \dfrac{1}{\sqrt{L}}[\exp(\mathrm{i}ky) + \exp(-\mathrm{i}\theta_k - \mathrm{i}ky)] & \text{if} \quad k \in \mathcal{K}^f \\ \left[\dfrac{\mu(\kappa-2)}{1-(\mu-1)^L}\right]^{1/2} (-1)^y (\kappa-1)^{-y/2} & \text{if} \quad k = \pi \end{cases} \tag{37}$$

and

$$\lambda_k = \begin{cases} \dfrac{1}{\sqrt{\kappa-1}} + \sqrt{\kappa-1} & \text{if} \quad k = 0 \\ 2\cos(k) & \text{if} \quad k \in \mathcal{K}^f \\ -\dfrac{1}{\sqrt{\kappa-1}} - \sqrt{\kappa-1} & \text{if} \quad k = \pi \end{cases} \tag{38}$$

where $\exp(\mathrm{i}\theta_k)$ is given by

$$\exp(\mathrm{i}\theta_k) = -\frac{\lambda_k - \kappa\exp(-\mathrm{i}k)}{\lambda_k - \kappa\exp(\mathrm{i}k)} \tag{39}$$

and

$$\mathcal{K}^f = \left\{ \frac{\pi j}{L} \middle| j = 1, \ldots, L-1 \right\}. \tag{40}$$

Note that $\varphi_k(y)$ can be made real by taking out a phase factor. Thus, there are $L+1$ distinct values of λ_k, but if $k \to \pi - k$ then λ_k changes sign and hence there are only $(L+1)/2$ distinct eigenvalues, λ_k^2. Note, since L is assumed odd, λ_k is never equal to zero (if L is even this happens for $k = \pi/2$ which gives rise to a null space—a distracting complication we do not address). Since the dimension of the space upon which $\overset{e \cdot e}{\boldsymbol{T}}_1$ and $\overset{o \cdot o}{\boldsymbol{T}}_1$ act is $(L+1)/2$ we have a spanning set of eigenvectors labelled by the index set $\mathcal{K}_1$ which is given by

$$\mathcal{K}_1 = \left\{ k = \frac{\pi \ell}{L} \middle| \ell = 0, \ldots, \frac{L-1}{2} \right\}. \tag{41}$$

Thus we have satisfied the conditions of the BEO theorem and hence

$$\overset{=\mathcal{N}}{Z_t}(\boldsymbol{y}^i \to \mathbf{y}^f) = \det \| \overset{=\mathcal{S}}{Z_t}(y_\alpha^i \to y_\beta^f) \|_{\alpha,\beta=1,\ldots,N} \tag{42}$$

where

$$\overset{=\mathcal{S}}{Z_t}(y^i \to y^f) = v(y^i) \sum_{k \in \mathcal{K}_1} \overset{p}{\varphi}{}_k^{\mathrm{R}}(y^i) \lambda_k^t \overset{p'}{\varphi}{}_k^{\mathrm{L}*}(y^f). \tag{43}$$

5.2. *Evaluation of the one-walk partition function from the spectral decomposition*

For the sake of completeness (as it has not appeared previously) we briefly demonstrate how the expression (43) for the partition functions required can be evaluated. This method differs from that previously used to obtain a ‘constant term’ solution [1]. First note that the partition function is zero if $t + y^i + y^f$ is odd. For $t + y^i + y^f$ even, substitution from (36) gives

$$\overset{=\mathcal{S}}{Z_t}(y^i \to y^f) = \sum_{k \in \mathcal{K}_1} \varphi_k(y^i) \lambda_k^t \ \varphi_k^*(y^f) \tag{44}$$

and using (37),

$$\overset{=\mathcal{S}}{Z_t}(y^i \to y^f) = \frac{\mu(\kappa-2)}{1-(\mu-1)^L} \lambda_0^t (\kappa-1)^{-(y^i+y^f)/2} + \frac{1}{L} \sum_{k \in \mathcal{K}_1 - \{0\}} \lambda_k^t (Q_k + Q_{-k}) \tag{45}$$

where

$$Q_k = \mathrm{e}^{\mathrm{i}k(y^f - y^i)} + \mathrm{e}^{\mathrm{i}k(y^f + y^i)} \mathrm{e}^{\mathrm{i}\theta_k}. \tag{46}$$

Since $Q_{2\pi-k} = Q_{-k}$, $\lambda_{\pi-k}^t Q_{\pi-k} = (-1)^{t+y^i+y^f} \lambda_k^t Q_{-k}$ and $Q_0 = Q_\pi = 0$ the range of k values summed over can be extended to give,

$$\overset{=\mathcal{S}}{Z_t}(y^i \to y^f) = \frac{\mu(\kappa-2)\kappa^t}{1-(\mu-1)^L} (\kappa-1)^{-(t+y^i+y^f)/2} + \frac{1}{2L} \sum_{k \in \mathcal{K}^+} \lambda_k^t Q_k \tag{47}$$

where we have used (38) and

$$\mathcal{K}^+ = \left\{ \frac{\pi \ell}{L} \middle| \ell = 0, \ldots, 2L-1 \right\}. \tag{48}$$

Hence, if we now use the result that, for $\Delta \in \mathbb{Z}$,

$$\frac{1}{2L} \sum_{n=0}^{2L-1} \exp\left(\mathrm{i}\pi \frac{n}{L} \Delta \right) = \begin{cases} 1 & \text{if } \Delta \in \{0, \pm 2L, \pm 4L, \ldots\} \\ 0 & \text{otherwise} \end{cases} \tag{49}$$

we get, after expanding the denominator of $\mathrm{e}^{\mathrm{i}\theta_k}$ (valid for $|\kappa - 1| < 1$), the result (23).

Returning to (45), the summand is a continuous function if $\kappa \neq 2$ ($\exp(\mathrm{i}\theta_k)$ has a zero in the denominator for $\kappa = 2$), and thus in the limit $L \to \infty$ the sum becomes a Riemann integral which can be evaluated by residues to give (24). This can also be obtained directly from (23). Since (23) and (24) are Laurent polynomials in $\bar{\kappa}$ (since the binomial coefficients are assumed to vanish outside there natural domain of definition) the results are valid for all κ.

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